Fractional order differentiation by integration: an application to fractional linear systems

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Abstract: In this article, we propose a robust method to compute the output of a fractional linear system defined through a linear fractional differential equation (FDE) with time-varying coefficients, where the input can be noisy. We firstly introduce an estimator of the fractional derivative of an unknown signal, which is defined by an integral formula obtained by calculating the fractional derivative of a truncated Jacobi polynomial series expansion. We then approximate the FDE by applying to each fractional derivative this formal algebraic integral estimator. Consequently, the fractional derivatives of the solution are applied on the used Jacobi polynomials and then we need to identify the unknown coefficients of the truncated series expansion of the solution. Modulating functions method is used to estimate these coefficients by solving a linear system issued from the approximated FDE and some initial conditions. A numerical result is given to confirm the reliability of the proposed method.

Keywords: Differential equations; Differentiators; Parameter estimation.

1. INTRODUCTION

Fractional calculus were introduced in many fields of science and engineering long time ago. It was first developed by mathematicians in the middle of the ninetieth century [1]. During the past decades, fractional calculus has gained great interest in several applications [2], for instance in control engineering [3], signal processing [4], and finance [5], etc. Unlike classical differential equations, fractional differential equations can better describe some natural phenomena and especially some dynamical systems [6, 7]. This is due essentially to their memory and hereditary properties. Many studies have been done in the existence and uniqueness of solutions to fractional differential equations (see, e.g. [8, 9, 10]). In the case of disturbed source, the behavior of solution has been studied in [12]. However, it is in general difficult to find analytic solutions to fractional differential equations, reliable approximations and numerical techniques are then essential. One can distinguish two main categories of methods for solving fractional differential equations: frequency-domain methods and time-domain methods. Frequency-domain methods include the popular Laplace transform method [8] and Fourier transform method [9]. They usually provide analytical solutions for fractional differential equations. Concerning time-domain methods, an increasing number of numerical schemes are being developed for both linear and nonlinear fractional differential equations. In nonlinear case, these methods include Lubich's difference methods, Diethelm's method based on quadrature, Adams-Bashforth-Moulton method, Homotopy-perturbation method, Adomian's decomposition method (see [11]).

During the last few years, many authors applied function approximation theory to solve linear fractional differential equations with constant coefficients. Solutions were approximated by truncated series expansions involving different bases, such as orthogonal polynomials [13] and wavelets [14, 15], with unknown coefficients. Thus, the problem of solving a fractional differential equation was transformed into a problem of identifying the coefficients of an equation. Then, different methods were considered to solve this problem by solving a linear system of algebraic equations. These methods include operational matrix method [14], collocation method [15], and spectral tau method [13]. Unlike the spectral tau method used in [13], the operational matrix method in general uses polynomials to approximate the solutions of fractional differential equations, and the fractional derivatives or the fractional integrals of these solutions. But the fractional derivatives and the fractional integrals of a classical polynomial are not polynomials. Moreover, the collocation method interpolates source functions at collocation points which are usually not irregularly spaced. This is a constraint for many applications. Also, the method is not robust against noisy source functions. Very recently, Legendre polynomials and spectral tau method were used in [16] to solve fractional differential equations with time-varying coefficients. In this paper, we use Jacobi polynomials and the modulating function method to extend the methods used in [16].

There are practical situations where the source functions are measured and corrupted by an additive noise. However, none of the above methods has considered this case. In this paper, we are interested in the fractional order differentiation by integration with Jacobi polynomials. This method consists in estimating the fractional derivative of an unknown signal by an integral formula involving Jacobi polynomials and the noisy observation of the unknown signal [17, 18]. It generalizes the method of differentiation by integration [19, 20] from integer order to fractional order. Let us recall that the method of differentiation by integration with Jacobi polynomials is the generalization of the Lanczos generalized derivative [21] (p. 324) (1956) in noisy case. The differentiators proposed in [19] were originally introduced using the algebraic parametric techniques [22, 23, 24, 25, 26, 27, 28] which exhibited good robustness properties with respect to corrupting noises without the need of knowing their statistical properties [29].

One of the methods that have been proposed to solve parameter identification problems is the modulating function method. The latter was pioneered by Shinbrot [30] in 1957 to solve the parameter identification problem by using integral transformation. The idea was motivated by Laplace and Fourier transforms. Essentially, the use of modulating functions allows to transform a differential equation, involving input-output noisy signals on a specified time interval, into a sequence of algebraic equations. Moreover, it avoids the derivatives of the noisy signals and annihilates the effects of initial conditions, thus allows the direct use of noisy signals in the integral expression. Hence, this method takes advantage from the low-pass filtering property of modulating functions integrals [31, 29]. It was used to parameter identification for nonlinear systems, time-varying systems and noisy sinusoidal signals [32, 33, 31].

The aim of this paper is to propose a robust method to find the output of fractional linear systems, defined through linear fractional differential equations with time-varying coefficients, where the input can be noisy. For this purpose, we use the fractional order differentiation by integration with Jacobi polynomial and the modulating functions method. The calculus are made from the noisy input and some initial conditions. This paper is organized as follows: in Section 2, we recall the method of fractional order differentiation by integration. It is used in Section 3 to solve fractional differential equations together with the modulating function method. In Section 4, we apply the proposed method to an example. Finally, some conclusions and perspectives are given in Section 5.

2. FRACTIONAL ORDER DIFFERENTIATION BY INTEGRATION

In this section, we recall the main idea behind fractional order differentiation by integration [17, 18], which generalizes the method of differentiation by integration with Jacobi polynomials [19, 20] from the integer order to the fractional order. Before doing so, we recall the definitions and some useful properties of the Caputo fractional derivative and the Jacobi orthogonal polynomials, respectively.

2.1 Caputo fractional derivative

Unlike classical integer order derivatives, there are several definitions for fractional derivatives which are in general not equivalent with each others [8, 9]. In this paper, we use the Caputo fractional derivative.

Let $f \in \mathcal{C}^l(\mathbb{R})$ with $l \in \mathbb{N}^*$, where $\mathcal{C}^l(\mathbb{R})$ denotes the set of the l-times continuously differentiable functions defined on $\mathbb{R}$. Then, the Caputo fractional derivative (see [8] p. 79) of f is defined as follows: $\forall\, t \in \mathbb{R}_+^*$,

$$\mathrm{D}_t^\alpha f(\cdot) := \frac{1}{\Gamma(l-\alpha)} \int_0^t (t-\tau)^{l-\alpha-1} f^{(l)}(\tau)\, d\tau, \quad (1)$$

where $0 \le l-1 < \alpha \le l$, and $\Gamma(z) = \int_0^\infty \exp(-x)\, x^{z-1}\, dx$ is the Gamma function (see [34] p. 255). Hence, the Caputo fractional derivative of an n^{th} order polynomial $f(t) = t^n$ can be given as follows (see [8] p. 72): $\forall\, t \in \mathbb{R}_+$,

$$\mathrm{D}_t^\alpha f(\cdot) = \begin{cases} 0, & \text{if } n < \alpha, \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, t^{n-\alpha}, & \text{if } n \ge \alpha. \end{cases} \quad (2)$$

In the following are some useful proprieties of the Caputo fractional derivative: $\forall\, t \in \mathbb{R}_+^*$,

- Linearity (see [8] p. 91):
$$\mathrm{D}_t^\alpha \left\{\lambda_1 f_1(\cdot) + \lambda_2 f_2(\cdot)\right\} = \lambda_1 \mathrm{D}_t^\alpha f_1(\cdot) + \lambda_2 \mathrm{D}_t^\alpha f_2(\cdot), \quad (3)$$
- Scale change (see [2] p. 76):
$$\mathrm{D}_{\lambda\, t}^\alpha f(\cdot) = \frac{1}{\lambda^\alpha} \mathrm{D}_t^\alpha f(\lambda\, \cdot), \quad (4)$$

where $\lambda \in \mathbb{R}_+^*$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f, f_1, f_2 \in \mathcal{C}^l(\mathbb{R})$.

2.2 Jacobi orthogonal polynomials

The n^{th} order shifted Jacobi orthogonal polynomial (see [34] p. 775) can be defined on $[0,1]$ as follows:

$$P_n^{(\mu,\kappa)}(\tau) := \sum_{j=0}^{n} \binom{n+\mu}{j} \binom{n+\kappa}{n-j} (\tau-1)^{n-j}\, \tau^j, \quad (5)$$

where $\mu, \kappa \in]-1, +\infty[$, and $\binom{n+\mu}{j} = \frac{\Gamma(n+\mu+1)}{\Gamma(n+\mu-j+1)\,\Gamma(j+1)}$, $\binom{n+\kappa}{n-j} = \frac{\Gamma(n+\kappa+1)}{\Gamma(\kappa+j+1)\,\Gamma(n-j+1)}$. Let g_1 and g_2 be two continuous functions defined on $[0,1]$, then we define the following scalar product (see [34] p. 774):

$$\langle g_1(\cdot), g_2(\cdot)\rangle_{\mu,\kappa} = \int_0^1 w_{\mu,\kappa}(\tau)\, g_1(\tau)\, g_2(\tau)\, d\tau, \quad (6)$$

where $w_{\mu,\kappa}(\tau) = (1-\tau)^\mu \tau^\kappa$ is the associated weighted function. Thus, the norm of the Jacobi polynomial $P_n^{(\mu,\kappa)}$ is given by: $\|P_n^{(\mu,\kappa)}\|_{\mu,\kappa}^2 = \frac{\Gamma(\mu+n+1)\,\Gamma(\kappa+n+1)}{\Gamma(\mu+\kappa+n+1)\,\Gamma(n+1)\,(2n+\mu+\kappa+1)}$.

Then, by applying the linearity (3) to (5) and using (2), the Caputo fractional derivative of the Jacobi polynomial can be given in the following lemma.

Lemma 1. The α^{th} ($\alpha \in \mathbb{R}_+$) order derivative of the n^{th} order Jacobi orthogonal polynomial $P_n^{(\mu,\kappa)}$ defined in (5) is given as follows: $\forall\, \tau \in [0,1]$,

$$\mathrm{D}_\tau^\alpha P_n^{(\mu,\kappa)}(\cdot) = \begin{cases} 0, & \text{if } n < \alpha, \\ \displaystyle\sum_{j=0}^{n} \sum_{i=0}^{\min(n-j,n-l)} \frac{c_{\mu,\kappa,n,j,i}\,\Gamma(n-i+1)}{\Gamma(n-i+1-\alpha)}\,\tau^{n-i-\alpha}, & \text{if } n \geq \alpha, \end{cases}$$

where $c_{\mu,\kappa,n,j,i} = (-1)^i \binom{n+\mu}{j}\binom{n+\kappa}{n-j}\binom{n-j}{i}$, and $l \in \mathbb{N}^*$ with $0 \leq l-1 < \alpha \leq l$.

2.3 Fractional Jacobi differentiator

Let $y \in \mathcal{C}^l(\mathbb{R})$, and y^ϖ be a noisy observation of y on an interval $I = [0,h] \subset \mathbb{R}_+$:

$$\forall\, t \in I,\ y^\varpi(t) = y(t) + \varpi(t), \tag{7}$$

where ϖ is an integrable noise [1]. We are going to estimate the α^{th} order derivative of y using the observation y^ϖ.

Let us ignore the noise for a moment. By taking the Jacobi orthogonal series expansion of $y(h\cdot)$ ([36] p. 6), we have: $\forall\, \xi \in [0,1]$,

$$y(h\xi) = \sum_{i=0}^{+\infty} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y(h\cdot)\right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|_{\mu,\kappa}^2}\ P_i^{(\mu,\kappa)}(\xi). \tag{8}$$

We take the truncated Jacobi orthogonal series expansion of $y(h\cdot)$ to approximate y on I by the following N^{th} ($N \in \mathbb{N}$) order polynomial: $\forall\, \xi \in [0,1]$,

$$D_{h,\mu,\kappa,N}^{(0)} y(h\xi) := \sum_{i=0}^{N} \lambda_i^{(\mu,\kappa)}\, P_i^{(\mu,\kappa)}(\xi), \tag{9}$$

where $\lambda_i^{(\mu,\kappa)} = \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y(h\cdot)\right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|_{\mu,\kappa}^2}$. Then, we calculate the fractional derivative of the polynomial $D_{h,\mu,\kappa,N}^{(0)} y(\cdot)$ such that we can approximate the fractional derivative of y. We denote the α^{th} order derivative of $D_{h,\mu,\kappa,N}^{(0)} y(\cdot)$ by $D_{h,\mu,\kappa,N}^{(\alpha)} y(\cdot)$. By applying the scale change property (4) and the linearity (3), to (9), we obtain: $\forall\, \xi \in [0,1]$,

$$\begin{aligned} \mathrm{D}_{h\xi}^\alpha y(\cdot) \approx& D_{h,\mu,\kappa,N}^{(\alpha)} y(h\xi) \\ :=& \mathrm{D}_{h\xi}^\alpha \left\{ D_{h,\mu,\kappa,N}^{(0)} y(\cdot) \right\} \\ =& \frac{1}{h^\alpha} \mathrm{D}_\xi^\alpha \left\{ D_{h,\mu,\kappa,N}^{(0)} y(h\cdot) \right\} \\ =& \frac{1}{h^\alpha} \sum_{i=0}^{N} \lambda_i^{(\mu,\kappa)}\, \mathrm{D}_\xi^\alpha P_i^{(\mu,\kappa)}(\cdot). \end{aligned} \tag{10}$$

Consequently, by substituting y by its noisy observation y^ϖ in (10) we obtain the following fractional order differentiator: $\forall\, \xi \in [0,1]$,

$$\mathrm{D}_{h\xi}^\alpha y(\cdot) \approx D_{h,\mu,\kappa,N}^{(\alpha)} y^\varpi(h\xi) := \frac{1}{h^\alpha} \sum_{i=0}^{N} \tilde{\lambda}_i^{(\mu,\kappa)}\, \mathrm{D}_\xi^\alpha P_i^{(\mu,\kappa)}(\cdot), \tag{11}$$

where $\tilde{\lambda}_i^{(\mu,\kappa)} = \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y^\varpi(h\cdot)\right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|_{\mu,\kappa}^2}$. By expanding the scalar product $\langle \cdot,\cdot\rangle_{\mu,\kappa}$, we get the following integral form for $D_{h,\mu,\kappa,N}^{(\alpha)} y^\varpi(h\xi)$: $\forall\, \xi \in [0,1]$,

$$D_{h,\mu,\kappa,N}^{(\alpha)} y^\varpi(h\xi) = \frac{1}{h^\alpha} \int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,\xi)\, y^\varpi(h\tau)\, d\tau, \tag{12}$$

where $Q_{\mu,\kappa,\alpha,N}(\tau,\xi) = w_{\mu,\kappa}(\tau) \sum_{i=0}^{N} \frac{P_i^{(\mu,\kappa)}(\tau)}{\|P_i^{(\mu,\kappa)}\|_{\mu,\kappa}^2}\, \mathrm{D}_\xi^\alpha P_i^{(\mu,\kappa)}(\cdot)$.

According to (12), the differentiator $D_{h,\mu,\kappa,N}^{(\alpha)} y^\varpi(h\xi)$ uses an integral involving Jacobi polynomials to estimate the fractional order derivative of a noisy signal. We call this differentiator *fractional Jacobi differentiator* as in [17, 18], and this method *fractional order differentiation by integration*.

The estimation error for the fractional Jacobi differentiator in the noisy case can be divided into two sources: the truncated term error which is due to the truncated terms in the Jacobi orthogonal series expansion of $y(h\cdot)$, and the noise error contribution which is due to the noise ϖ. These errors have been studied in [18]. Moreover, it has been also showed in [18] how to choose the design parameters N, κ and μ for the fractional Jacobi differentiator so as to reduce these errors.

3. APPLICATION TO FRACTIONAL LINEAR SYSTEMS

3.1 Fractional linear systems

In this section, we consider a fractional linear system defined by a **F**ractional **D**ifferential **E**quation (FDE) in the following form: $\forall\, t \in I = [0,h] \subset \mathbb{R}_+$,

$$\sum_{i=0}^{L} a_i(t)\, \mathrm{D}_t^{\alpha_i} y(\cdot) = \sum_{j=0}^{M} b_j(t)\, \mathrm{D}_t^{\beta_j} u(\cdot), \tag{13}$$

where y is the output, u is the input, $L, M \in \mathbb{N}$, $a_i \neq 0$, $b_j \neq 0$ are time-varying coefficients and assumed to be known. We suppose that $\alpha_i, \beta_j \in \mathbb{R}_+$ are such that $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_L$ and $0 \leq \beta_1 < \beta_2 < \cdots < \beta_M$.

Let u^ϖ be a noisy observation of u on I. The goal is to find an approximation of the output y by solving the fractional differential equation defined in (13) with the noisy input u^ϖ and the following initial conditions:

$$y^{(i)}(0) = d_i \in \mathbb{R}, \quad \text{for} \quad i = 0, \ldots, \lceil \alpha_L \rceil - 1, \tag{14}$$

where $\lceil \alpha_L \rceil$ denotes the smallest integer greater than or equal to α_L. The existence and uniqueness of solution of the initial value problem (13)-(14) have been studied in [10] in noise-free case. We are going to numerically solve this problem by using the fractional Jacobi differentiator and the modulation functions method in the noisy case.

3.2 Application of fractional Jacobi differentiator

In order to solve the initial value problem (13)-(14), we use the polynomial defined in (9) to approximate the solution

[1] More generally, the noise is a stochastic process, which is integrable in the sense of convergence in mean square (see [35]).

y. Moreover, the fractional order derivatives of y can be approximated by using the fractional Jacobi differentiator given in (10). On the one hand, the left side of the FDE (13) can be approximated as follows: $\forall\, \xi \in [0,1]$,

$$\begin{aligned}\sum_{i=0}^{L} a_i(h\xi)\, \mathrm{D}_{h\xi}^{\alpha_i} y(\cdot) &\approx \sum_{i=0}^{L} a_i(h\xi)\, D_{h,\mu,\kappa,N}^{(\alpha_i)} y(h\xi) \\ &= \sum_{i=0}^{L} \frac{a_i(h\xi)}{h^{\alpha_i}} \sum_{j=0}^{N} \lambda_j^{(\mu,\kappa)} q_{\mu,\kappa,j}^{(\alpha_i)}(\xi) \\ &= \sum_{j=0}^{N} \lambda_j^{(\mu,\kappa)} \sum_{i=0}^{L} \frac{a_i(h\xi)}{h^{\alpha_i}} q_{\mu,\kappa,j}^{(\alpha_i)}(\xi),\end{aligned} \tag{15}$$

where $q_{\mu,\kappa,j}^{(\alpha_i)}(\xi) = \mathrm{D}_{\xi}^{\alpha_i} P_j^{(\mu,\kappa)}(\cdot)$. On the other hand, we denote the right side of the FDE (13) by $U(\xi) = \sum_{j=0}^{M} b_j(h\xi)\, \mathrm{D}_{h\xi}^{\beta_j} u(\cdot)$. Then, $U(\cdot)$ can be estimated by using the fractional Jacobi differentiator given in (12) and the noisy input u^{ϖ}. Hence, the FDE (13) is approximated by the following equation:

$$\forall\, \xi \in [0,1], \;\; \sum_{j=0}^{N} \lambda_j^{(\mu,\kappa)}\, Q_j^{(\mu,\kappa)}(\xi) = \tilde{U}(\xi), \tag{16}$$

where $Q_j^{(\mu,\kappa)}(\xi) = \sum_{i=0}^{L} \frac{a_i(h\xi)}{h^{\alpha_i}} q_{\mu,\kappa,j}^{(\alpha_i)}(\xi)$, and $\tilde{U}(\cdot)$ is an estimation of $U(\cdot)$. Consequently, the problem of solving the FDE (13) is transformed into a problem of identification of unknown coefficients $\lambda_j^{(\mu,\kappa)}$, for $j = 0, 1, \ldots, N$. We are going to apply the modulating functions method to solve this problem.

3.3 Modulating function method for FDEs

Let us define the following subset of the fractional orders in (13): $B := \{\beta_j; \beta_j \in \mathbb{N}, j = 0, \ldots, M\}$, and $\beta_* = \max_{\beta_j \in B} \beta_j$. If $B = \emptyset$, then $\beta_* = 0$. Then, we take a class of functions $\{g_i\}_{i=\lceil \alpha_L \rceil}^{N+1}$ which satisfy the following conditions:

$$\begin{cases} g_i \in \mathcal{C}^{\beta_*}([0,1]), \\ g_i^{(j)}(0) = g_i^{(j)}(1) = 0, \;\forall\, j = 0, 1, \ldots, \beta_* - 1. \end{cases} \tag{17}$$

These functions are called *modulating functions* [32].

By multiplying equation (16) by each function g_i and by integrating them on $[0,1]$, we then obtain the following system:

$$\sum_{j=0}^{N} \lambda_j^{(\mu,\kappa)} \int_0^1 g_i(\xi)\, Q_j^{(\mu,\kappa)}(\xi)\, d\xi = \int_0^1 g_i(\xi)\, \tilde{U}(\xi)\, d\xi, \tag{18}$$

for $i = \lceil \alpha_L \rceil, \ldots, N$. Then, the right side of (18) is calculated in the following way:

- If $\beta_j \in \mathbb{R}_+ \backslash \mathbb{N}$ in (13), then we estimate $\mathrm{D}_{h\xi}^{\beta_j} u(\cdot)$ by the fractional Jacobi differentiator $D_{h,\mu,\kappa,N}^{(\beta_j)} u^{\varpi}(h\xi)$ in $\tilde{U}(\xi)$.
- If $\beta_j \in \mathbb{N}$ in (13), then by applying β_j times integration by parts and the property (17) which annihilates the boundary values, we obtain:

$$\begin{aligned}&\int_0^1 g_i(\xi)\, b_j(h\xi)\, u^{(\beta_j)}(h\xi)\, d\xi \\ &\quad = \frac{(-1)^{\beta_j}}{h^{\beta_j}} \int_0^1 \frac{d^{\beta_j}}{d\xi^{\beta_j}} \left\{ g_i(\xi)\, b_j(h\xi) \right\} u(h\xi)\, d\xi.\end{aligned} \tag{19}$$

Hence, by using the noisy input u^{ϖ} we get:

$$\begin{aligned}&\int_0^1 g_i(\xi)\, b_j(h\xi)\, u^{(\beta_j)}(h\xi)\, d\xi \\ &\quad \approx \frac{(-1)^{\beta_j}}{h^{\beta_j}} \int_0^1 \frac{d^{\beta_j}}{d\xi^{\beta_j}} \left\{ g_i(\xi)\, b_j(h\xi) \right\} u^{\varpi}(h\xi)\, d\xi.\end{aligned} \tag{20}$$

Thus, we avoid the estimation of the integer order derivative of the noisy signal u^{ϖ} in (18). The estimation error can be reduced (see [35, 37, 38] for more details on the error analysis for the integer order differentiation by integration method).

Consequently, due to the low-pass filtering property of the fractional Jacobi differentiator and the modulating function method, the noise effect in the right side of (18) can be reduced. Now, we take the values of the integer order derivatives of the approximation polynomial defined in (9) at $\xi = 0$ to estimate the initial conditions. By applying similar calculations than those developed for (10) we then get:

$$\sum_{j=0}^{N} \lambda_j^{(\mu,\kappa)} \frac{q_{\mu,\kappa,j}^{(i)}(0)}{h^i} = d_i, \text{ for } i = 0, \ldots, \lceil \alpha_L \rceil - 1. \tag{21}$$

Consequently, by using (18) and (21), we can solve the following linear system to estimate the coefficients $\lambda_i^{(\mu,\kappa)}$:

$$M_N^{(\mu,\kappa)} \begin{pmatrix} \lambda_0^{(\mu,\kappa)} \\ \vdots \\ \lambda_N^{(\mu,\kappa)} \end{pmatrix} = \begin{pmatrix} d_0 \\ \vdots \\ d_N \end{pmatrix}, \tag{22}$$

where $d_i = \int_0^1 g_i(\xi)\, \tilde{U}(\xi)\, d\xi$, for $i = \lceil \alpha_L \rceil, \ldots, N$, and $M_N^{(\mu,\kappa)}(i,j) =$

$$\begin{cases} \dfrac{q_{\mu,\kappa,j}^{(i)}(0)}{h^i}, & \text{for } 0 \le i \le \lceil \alpha_L \rceil - 1, 0 \le j \le M, \\ \displaystyle\int_0^1 g_i(\xi)\, Q_j^{(\mu,\kappa)}(\xi)\, d\xi, & \text{for } \lceil \alpha_L \rceil \le i \le N, 0 \le j \le M. \end{cases}$$

4. NUMERICAL RESULTS

In order to demonstrate the reliability of our method, we consider the following FDE (see [16]):

$$\begin{cases} \displaystyle\sum_{i=0}^{4} a_i(t)\, \mathrm{D}_t^{\alpha_i} y(\cdot) = f(t), \\ y(0) = 2, \; y'(0) = 0, \end{cases} \tag{23}$$

where $a_0(t) = t^{\frac{1}{5}}$, $a_1(t) = t^{\frac{1}{4}}$, $a_2(t) = t^{\frac{1}{3}}$, $a_3(t) = t^{\frac{1}{2}}$, $a_4(t) \equiv 1$, $\alpha_0 = 0$, $\alpha_1 = 0.333$, $\alpha_2 = 1$, $\alpha_3 = 1.234$, $\alpha_4 = 2$, and $y(t) = 2 - \frac{1}{2}t^2$. We need to choose an input such that f satisfies (23). The choice of this input is not unique. In this example, we take $u(\cdot) = y(\cdot)$ such that $f(t) = \sum_{j=0}^{4} a_j(t)\, \mathrm{D}_t^{\alpha_j} u(\cdot)$. Then, we propose to estimate the output y by using the discrete noisy observation of the input: $u^{\varpi}(t_i) = u(t_i) + \sigma \varpi(t_i)$, where $t_i = iT_s \in [0,1]$,

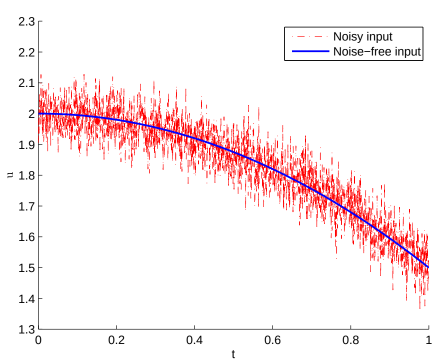

Fig. 1. The noise-free input and its noisy observation.

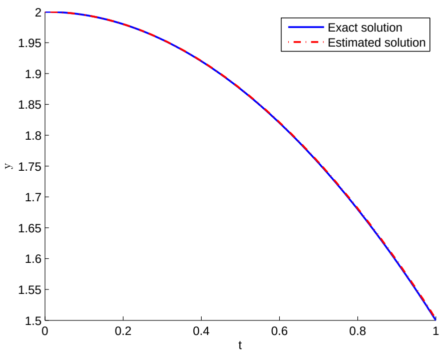

Fig. 2. The exact output and the output estimated.

$T_s = \frac{1}{2000}$, for $i = 0, \cdots, 2000$, the noise $\varpi(t_i)$ is simulated from a zero-mean white Gaussian *iid* sequence, and $\sigma \in \mathbb{R}_+^*$ is adjusted in such a way that the signal-to-noise ratio $SNR = 10 \log_{10} \left(\frac{\sum |u^{\varpi}(t_i)|^2}{\sum |\sigma \varpi(t_i)|^2} \right)$ is equal to $SNR = 30\text{dB}$. We can see this discrete noisy input in Figure 1.

Since the noisy input is given in the discrete case, we apply the trapezoidal rule to approximate the integrals in (12) and (18). This produces a numerical error in the estimated output. Since y and u are two polynomials of degree 2, we take $N = 2$ in the fractional Jacobi differentiators. Thus, there is no truncated term errors in the used fractional Jacobi differentiators. Moreover, we take $\mu = \kappa = 0$ and the following modulating function: $g_2(\xi) = \xi^3(1-\xi)^7$ for $\xi \in [0, 1]$. The estimation error in the estimated output can be divided into two sources: the numerical error and the noise error contribution. Since y is a polynomial, this estimation error can also be a polynomial. We can see the estimated output and the corresponding estimation error in Figure 2 and Figure 3.

5. CONCLUSION

In this paper, we have proposed a method to determine the output of a fractional linear system from some initial conditions, where the input can be noisy. The system is defined through a linear fractional differential equation with time-varying coefficients. An integration-based fractional order differentiation method has been used to estimate the fractional derivative of noisy signals. It has been shown that this method exhibits good robustness properties with respect to a corrupting noise. We have also shown that with this method, the problem of solving the linear fractional differential equation becomes a problem of identifying parameters. We have used modulating function method for the identification of the parameters which is suitable for time-varying linear systems involving input-output noisy signals. Numerical results confirms the efficiency of the proposed method. In order to reduce the estimation error with appropriate modulating functions, we will think to give some error analysis in a future work. Questions related to observability and parameter identification for a fractional linear system will also be considered with such approach.

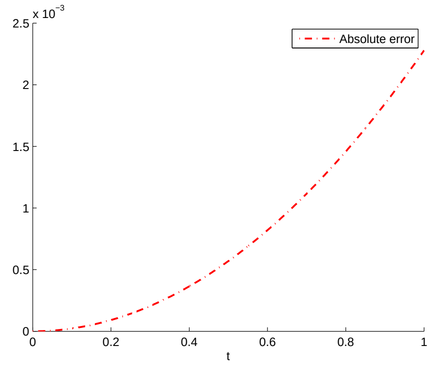

Fig. 3. Absolute estimation error.

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